Inhomogeneous Superconductivity in Comb-Shaped Josephson Junction Networks

P. Sodano[1,2], A. Trombettoni[2]
[1] *Progetto Lagrange, Fondazione C.R.T. e Fondazione I.S.I. c/o Dipartimento di Fisica-Politecnico di Torino, Corso Duca degli Abruzzi 24, Torino, I-10124, Italy*
[2] *Dipartimento di Fisica and Sezione I.N.F.N., Università di Perugia, Via A. Pascoli, Perugia, I-06123, Italy*

P.Silvestrini[3,4], R. Russo[3,4], and B. Ruggiero[4,3]
[3] *Dipartimento d'Ingegneria dell'Informazione, Seconda Universitá di Napoli, Aversa, Italy*
[4] *Istituto di Cibernetica "E. Caianiello" del CNR, Pozzuoli, Italy*

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We show that some of the Josephson couplings of junctions arranged to form an inhomogeneous network undergo a non-perturbative renormalization provided that the network's connectivity is pertinently chosen. As a result, the zero-voltage Josephson critical currents I_c turn out to be enhanced along directions selected by the network's topology. This renormalization effect is possible only on graphs whose adjacency matrix admits an hidden spectrum (i.e. a set of localized states disappearing in the thermodynamic limit). We provide a theoretical and experimental study of this effect by comparing the superconducting behavior of a comb-shaped Josephson junction network and a linear chain made with the same junctions: we show that the Josephson critical currents of the junctions located on the comb's backbone are bigger than the ones of the junctions located on the chain. Our theoretical analysis, based on a discrete version of the Bogoliubov-de Gennes equation, leads to results which are in good quantitative agreement with experimental results.

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It is a common belief that Josephson Junction Networks (JJN) may be regarded as the prototype of a complex physical system with a variety of interesting physical behaviors, adjustable acting only on a few external parameters and, by means of the modern fabrication technologies, also on the building topology and geometry of the array [1]. It is now possible to build arrays with very small junctions [2] to detect effects due to single electrons in a range of temperatures related to another relevant energy scale: the charging energy. The possibility of controlling experimentally the competition between Josephson and charging energies makes JJNs useful devices to investigate quantum phase transitions [3], or to model the physical properties of some real materials like granular superconductors [4]. Many of the results valid for JJNs are shared by cold atoms in optical lattices [5] since, in these systems, bosonic Josephson junctions and arrays may be rather easily realized [6]; in addition, Josephson networks and devices pave a very promising avenue to the quantum engineering of states relevant for quantum computing [7].

Inhomogeneous superconducting networks have been studied [8] mainly to provide a better understanding of the properties of well controlled disordered granular superconductors [9]. The appealing perspective to realize devices for the manipulation of quantum information recently stimulated the analysis of inhomogeneous planar JJNs with non conventional connectivity [10], engineered to sustain a topologically ordered ground state [11]. Transport measurements on superconducting wire networks evidenced - in a pure system with non-dispersive eigenstates- interesting anomalies of the network critical current induced by the interplay between the network's geometry and topology and an externally applied magnetic field [12]; more recently, the theoretical analysis of rhombi chains has evidenced the exciting possibility of being able to detect $4e$ superconductivity through measurements of the supercurrent in presence of a pertinent external magnetic field [13].

In this Letter we show that, even in absence of an externally applied magnetic field, a JJN fabricated on a pertinent graph [14] may support anomalous behaviors of the Josephson critical currents, which are induced by a non-perturbative renormalization of some of the Josephson couplings of the array. Our analysis clearly evidences that this renormalization is only attainable for the class of graphs, whose adjacency matrix supports an hidden spectrum [14,15]; thus, our findings are not generic to any inhomogeneous JJN. For instance, in absence of an external magnetic field, the networks analyzed in [8–10,12,13] should not give rise to any of the anomalous behaviors of I_c discussed in this paper.

In the following, we provide a theoretical and experimental study of the behavior of the Josephson critical currents measurable in a comb-shaped JJN made of Nb grains located at the vertices of a "comb" graph and linked by Josephson junctions [see Fig.1]. We compare our results with those obtained for a linear Josephson junction chain fabricated with the same junctions. Since one may regard the backbone of a comb graph as a decorated chain, it appears natural to compare its superconducting properties with those of a linear chain since the latter is the simplest network with euclidean dimension one. The result of this comparison shows that the

Josephson critical currents of the junctions located on the comb's backbone are sensibly bigger than the ones of the junctions located on the chain.

Another way to look at a comb-shaped JJN is to regard it as a linear chain immersed in an environment mimicked by the addition of the fingers [16]. As in many Josephson devices one should then expect that the nominal value of the Josephson energy E_J of the junctions in the array gets renormalized by the interaction with the environment. This situation is often analyzed using either the Caldeira-Leggett [17] or the electromagnetic environment [18] models. In these approaches one usually assumes that the *effective* boundary conditions for the quantum fluctuations of the environment modes do not depend on the Josephson couplings or on the network's topology: while this assumption is perfectly legitimate for weak environmental fluctuations, better care should be used if these fluctuations are strong as it may well happen for one dimensional JJNs. A simple paradigmatic example of a non perturbative renormalization of Josephson couplings is given by the simple inhomogeneous one-dimensional array analyzed in [19,20], where the source of inhomogeneity is given by putting on a site of the linear chain a *test* junction with a different nominal value of the Josephson coupling E_J. In the sequel we show that, for a comb-shaped JJN, the Josephson couplings on the backbone get renormalized. Our explicit computation shows that this renormalization is indeed non perturbative since the peculiar connectivity of a comb modifies the spectrum of quantum modes living on linear chains by the (obviously non-perturbative) addition of an infinite set of localized states, which disappear in the thermodynamic limit (the hidden spectrum): adding the fingers to a backbone chain is, in fact, a topological *operation* since it amounts to a non trivial change of boundary conditions for the Josephson linear chain. In a different context, the interplay between an hidden spectrum and a change in boundary conditions has been recently used in [21].

We use the lattice Bogoliubov-de Gennes (LBdG) equations [22] to compare the properties exhibited by Josephson linear chains and comb-shaped Josephson networks. Using the eigenfunctions of the LBdG equations, a self-consistent computation yields for both systems the gap function, the chemical potential and the quasi particle spectrum. We show that, for a linear chain, the superconducting gap and critical temperature satisfy to the well-known BCS equations and that, on the backbone of a comb JJN, the BCS equations are satisfied with a renormalized value of the Josephson energy. Then, we compute the zero-voltage Josephson critical currents I_c on the comb's backbone and compare our results for I_c with the outcomes of experimental measurements: our computation not only confirms with good accuracy the experimental results of [23], but is also in good agreement with new data obtained at temperatures closer to the critical temperature for the onset of superconductivity in Nb grains. The new data are shown in Fig.2.

To obtain a discrete version of the BdG equations suitable to describe the JJNs fabricated in [23], we make the ansatz that the eigenfunctions of the continuous BdG equations [22] may be written in a tight binding form as $u_\alpha(\vec{r}) = \sum_i u_\alpha(i)\phi_i(\vec{r})$ and $v_\alpha(\vec{r}) = \sum_i v_\alpha(i)\phi_i(\vec{r})$; i labels the position of a superconducting island while the contribution of the electronic states participating to superconductivity in a given island is effectively described by a field $\phi_i(\vec{r})$, whose specific form depends on the geometry of the islands and on the fabrication parameters of the connecting junctions. The LBdG equations then read

$$\in_\alpha u_\alpha(i) = \sum_j \epsilon_{ij} u_\alpha(j) + \Delta(i) v_\alpha(i) \tag{1}$$

$$\in_\alpha v_\alpha(i) = -\sum_j \epsilon_{ij} v_\alpha(j) + \Delta^*(i) u_\alpha(i). \tag{2}$$

where u_α and v_α satisfy to $\sum_i \left[|u_\alpha(i)|^2 + |v_\alpha(i)|^2\right] = 1$. The matrix ϵ_{ij} is defined by $\epsilon_{ij} = -tA_{ij} + U(i)\delta_{ij} - \tilde{\mu}\delta_{ij}$, with A_{ij} being the adjacency matrix of the network [24], $\tilde{\mu} = \mu - \int d\vec{r}\phi_i(\vec{r})\left(-\hbar^2\nabla^2/2m\right)\phi_i(\vec{r})$ and $t = -\int d\vec{r}\phi_i(\vec{r})[-\hbar^2\nabla^2/2m + U_0(\vec{r})]\phi_j(\vec{r}) \approx E_J$. $E_J = (\hbar/2e)I_c$ is the nominal value of the Josephson energy of all the junctions in the network while I_c is the unrenormalized zero-voltage Josephson critical current of each junction. $U_0(\vec{r})$ mimics the effects of the barrier between the superconducting islands. Self-consistency requires $\Delta(i) = \tilde{\mathcal{V}} \sum_\alpha u_\alpha(i) v^*_\alpha(i) \tanh\left(\frac{\beta}{2} \in_\alpha\right)$ and $U(i) = -\tilde{\mathcal{V}} \sum_\alpha \left[|u_\alpha(i)|^2 f_\alpha + |v_\alpha(i)|^2 (1 - f_\alpha)\right]$, where $\tilde{\mathcal{V}} \equiv \mathcal{V}\phi^2(\vec{r} = \vec{r}_i)$ is assumed to be independent on i. Topology is encoded in the term $-tA_{ij}$ appearing in the definition of the matrix ϵ_{ij}, while the specific values of t and $\tilde{\mathcal{V}}$ depend - as a result of our ansatz on the form of the eigenfunctions of the BdG equations- only on the $\phi_i(\vec{r})$.

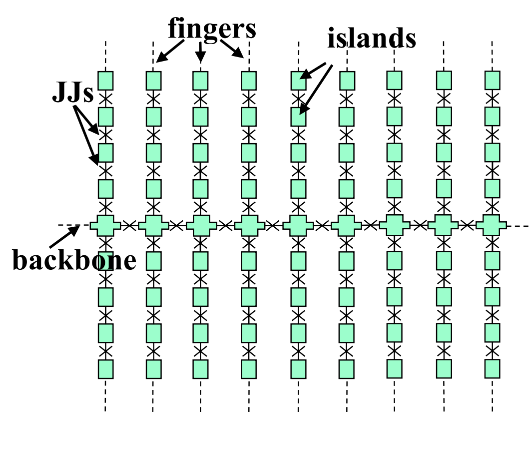

FIG. 1. Schematic drawing of a comb array. The superconducting islands (full box) are connected in series to each other through Josephson junctions (JJs). The finger arrays are connected to each other only through JJs to the central islands forming the backbone array.

To justify the assumptions involved in the derivation of Eqs.(1)- (2), we observe that, for the JJNs described in [23], capacitive (inter islands and with a ground) effects are negligible, that the total number of electrons on the island $\mathcal{N}$ is much larger than the number of electrons tunneling through the Josephson junction and that all islands contain approximately the same $\mathcal{N}$ ($\mathcal{N}(i) \equiv \mathcal{N}$). Furthermore, the islands are big enough to support the same superconducting gap of the Nb bulk material. As a result one may require $\phi_i(\vec{r})$ to be position-independent on each island except for a small region near the junction and to be the same on each island with a normalization given by $\int d\vec{r}\phi_i(\vec{r})\phi_i(\vec{r}) = \mathcal{N}(i) \equiv \mathcal{N}$ and $\int d\vec{r}\phi_i(\vec{r})\phi_j(\vec{r}) \approx 0$ for $i \neq j$. In our derivations we put $\mathcal{N} \equiv 1$.

For a linear array, the LBdG may be readily solved leading to an uniform potential $U(i) \equiv U_c$ and an uniform pair potential $\Delta(i) \equiv \Delta_c$. From the eigenvalue equation $-E_J \sum_j A_{ij}\psi_k(j) = e_k\psi_k(i)$, one gets $e_k = -2E_J \cos k$: it follows $\in_k = \sqrt{\Delta_c^2 + E_k^2}$ with $E_k = e_k + U_c - \tilde{\mu}$. The BCS-like behaviour is obtained when $E_k = e_k$, which happens since $U_c = 0$ and $\mu = E_F$. When $\Delta_c/E_J \ll 1$, for $T = 0$, one gets $\Delta_c(T = 0) = 8E_J e^{-2\pi E_J/\tilde{\mathcal{V}}}$, while, for $T = T_c$ (i.e., $\Delta_c(T = T_c) = 0$), one obtains $k_B T_c = \mathcal{C} E_J e^{-2\pi E_J/\tilde{\mathcal{V}}}$, with $\mathcal{C} = 4.54$. It is comforting that the assumptions on which our approach is based lead, for the chain, to results having the same functional form of the well-known BCS formulas for the gap at $T = 0$ (i.e., $\Delta(T = 0) = 2\hbar\omega_D e^{-1/n(0)V_{BCS}}$) and the BCS critical temperature (i.e., $k_B T_c = 1.14\hbar\omega_D e^{-1/n(0)V_{BCS}}$), provided that $n(0)V_{BCS} \ll 1$ [22]: in addition, one gets also $\Delta_c(T = 0)/k_B T_c = 8/\mathcal{C} \approx 1.76$.

Measurements on a chain made with Nb grains yield $T_c \approx 8.8K$ and $\Delta_c(T = 0) \approx 1.4meV \approx k_B \cdot 15.9K$; furthermore, in the experimental setup described in [23] it is $I_c \approx 18\mu A$. The parameters E_J and $\tilde{\mathcal{V}}$, determined from the BCS equation yielding the chain's critical temperature, are then given by $E_J \approx k_B \cdot 430K$ and $\tilde{\mathcal{V}}/E_J = 1.185$. In Fig.2 we plot for several temperatures the measured I_c (circles) and the critical currents obtained inserting $\Delta_c(T)$ in the well-known Ambegaokar-Baratoff expression [25] for the zero-voltage Josephson current (lower solid curve): the agreement is excellent.

For a comb network with $N \times N$ islands (see Fig.1), one finds a solution of the LBdG equations (1)- (2) where both the Hartree-Fock potential $U(i)$ and the gap function $\Delta(i)$ are position dependent. We denote the islands by (x, y), x labeling the finger and $|y|$ the distance from the backbone, expressed in lattice units. The eigenvalue equation $-E_J \sum_j A_{ij}\psi_\alpha(j) = e_\alpha\psi_\alpha(i)$, admits [24], in addition to a set of delocalized states with energies ranging from $-2E_J$ to $2E_J$, a localized ground-state $\psi_0 = (C_0/\sqrt{N})e^{-y/\xi}$, corresponding to the eigenvalue $e_0 = -2\sqrt{2}E_J$ ($C_0^2 = 1/\sqrt{2}$ and ξ given by $\sinh(1/\xi) = 1$) and an hidden spectrum made of other eigenstates localized around the backbone [24]. For a crude analytical estimate, one may require that, away from the backbone, the fingers may be regarded as a linear chain with uniform potentials (i.e., $\Delta(i) = \Delta_c$ and $U(i) = U_c$). To get then coupled equations for Δ_b, Δ_c, U_b, and U_c, one writes the LBdG equations (1)-(2) on a backbone's grain i. We set $u_\alpha(i) = U_\alpha\psi_\alpha(i)$ and $v_\alpha(i) = V_\alpha\psi_\alpha(i)$, with $U_\alpha^2 + V_\alpha^2 = 1$. The self-consistency equation for U implies that, at $T = 0$, $U_b \approx U_c - \frac{\tilde{\mathcal{V}}C_0^2}{2}$; upon requiring $\tilde{\mu} \approx U_b$ one immediately sees that, due to the localized modes in the fermionic spectrum, the chemical potential on the comb's backbone is smaller than the one measured on the chain.

By substituting the wavefunctions of the eigenstates of the hidden spectrum [24] in Eqs.(1)-(2) and using $\tilde{\mu} \approx U_b$ one gets

$$\Delta_b = \Delta_c + \frac{\Delta_b\tilde{\mathcal{V}}}{\pi} \cdot \int_0^{\pi/2} dk \frac{\cos k}{\in_k \sqrt{1 + \cos^2 k}} \cdot \tanh\left(\frac{\beta}{2} \in_k\right). \tag{3}$$

where $\in_k = \sqrt{\Delta_b^2 + 4E_J^2(1 + \cos^2 k)}$. The hidden spectrum eigenstates contribute to the gap function Δ_b through the second term in the rhs of Eq.(3): without them, Δ_b equals Δ_c.

When $E_J \gg \Delta_b, \Delta_c$, Eq.(3), at $T = 0$, yields $\frac{\Delta_b(T=0)}{\Delta_c(T=0)} = \frac{1}{1 - \frac{\eta_c\tilde{\mathcal{V}}}{2\pi E_J}} \equiv \mathcal{K}$ where $\eta_c \equiv (1/\sqrt{2}) \log\left(1 + \sqrt{2}\right)$. Furthermore, at low temperatures, $\Delta_b(T)/\Delta_c(T) \approx \Delta_b(T = 0)/\Delta_c(T = 0)$. Using the parameters E_J and $\tilde{\mathcal{V}}$ obtained from the measurements carried on the JJ chain, for a JJ comb one gets $\mathcal{K} \approx 1.13$.

Upon requiring that, as for the linear chain, the $T = 0$ backbone's gap function has a BCS like functional form, i.e. $\Delta_b(T = 0) = 8\bar{E}_J e^{-2\pi\bar{E}_J/\bar{\tilde{\mathcal{V}}}}$, with $\bar{E}_J$ and $\bar{\tilde{\mathcal{V}}}$ the renormalized Josephson energy and the renormalized interaction term, one is able to estimate the renormalization of the Josephson coupling within the LBdG approach. Namely, one has,

$$\bar{E}_J = \mathcal{K}E_J; \qquad \bar{\tilde{\mathcal{V}}} = \mathcal{K}\tilde{\mathcal{V}}, \tag{4}$$

which embodies the effects of the hidden spectrum on the Josephson couplings.

In Fig.2 we plot, as a function of the normalized temperature, the values of I_c measured with the methods described in [23] (squares) and the values of I_c obtained from the Ambegaokar-Baratoff formula using both the renormalized coupling given by Eq.(4) and the gap function along the backbone for the comb-like JJN studied in [23] (solid curve): the agreement between theory and experiments is very good at low temperatures, while the theory gives a slight overestimate at higher temperature.

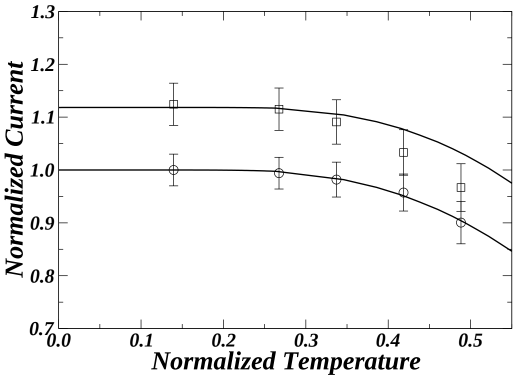

FIG. 2. Critical currents (in units of the critical current on the reference chain at $T = 1.2K$) as a function of T/T_c for the backbone and the chain. The solid lines are the estimated critical currents for the backbone (top) and the chain (bottom). Circles (squares): experimental values for the chain (backbone).

We showed that a non perturbative (i.e. induced by the states of the hidden spectrum) renormalization of some of the Josephson couplings of a comb-shaped JJN is responsible for the observed enhancement of I_c of the Josephson junctions located along the comb's backbone. The key assumption in our derivation is that the eigenfunctions of the BdG equations may be written in a tight binding form; once this assumption is made, one is able to derive Eqs. (1)- (2) and to account for all the dependence on the electronic states into the definition of the parameters E_J and $\tilde{\mathcal{V}}$, which, in this paper, we determined from the measurements carried on the linear chain. Our approach yields a value of the renormalized Josephson coupling of the junctions located on the comb's backbone in excellent agreement with the experimental results (see fig.2). We expect that similar phenomena happen for the class [15] of JJNs fabricated on graphs whose adjacency matrix supports an hidden spectrum.

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